On permanents of matrices over a commutative additively idempotent semiring

Yan Huang Haifeng Lian*

Department of mathematics, Fujian Agriculture and Forestry University, Fuzhou, 350002, China;

Email: xiaowuyav@126.com, lianhaif@qq.com

Abstract: Let R be a commutative additively idempotent semiring. In this paper, some properties and characterizations for permanents of matrices over R are established, and several inequalities for permanents are given. Also, the adjiont matrices of matriecs over R are considered. Partial results obtained in this paper generalize the corresponding ones on fuzzy matrices, on lattice matrices and on incline matrices.

Keywords: Permanent; adjoint matrix; additively idempotent semiring.

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1 Introduction

A *semiring* is an algebraic system $(R, +, \cdot)$ in which $(R, +)$ is an abelian monoid with identity element 0 and $(R, \cdot)$ is another monoid with identity element $1(\neq 0)$. In addition, operations $+$ and $\cdot$ are connected by distributivity and 0 annihilates R. A semiring is commutative if $ab = ba$ for all $a, b \in R$. A semiring R is called an *incline* if $a + 1 = 1$ for all $a \in R$. Every Boolean algebra, the fuzzy algebra $\mathbb{F} = ([0, 1], \vee, T)$, where $\vee = \max$ and T is a t-norm (for t-norm, see [7]), and any bounded distributive lattice are examples of inclines.

A semiring R is called *an additively idempotent semiring* if $a + a = a$ for all $a \in R$. Clearly, an incline is additively idempotent. Additively idempotent semirings are useful tools in diverse areas such as fuzzy set theory and decision analysis; data analysis and preference modeling; classical and non-classical path-finding problems in graphs; analysis and control of discrete-event systems (see [4]).

Permanent of a matrix made its first appearance in the famous memoirs of Binet[1]. Since then, a large number of works on permanent theory have been published (see [3, 6, 8, 9, 11, 13]). The *adjoin matrix* of $A \in M_n(R)$ is denoted by $\mathrm{adj}(A)$, which is defined as an $n \times n$ matrix whose (i, j)th entry is the permanent of $A(j|i)$, where $A(j|i) \in M_{n-1}(R)$ obtained from A by deleting the jth row and the ith column. Since the late 1980s, many authors have studied adjoint matrix over special cases of additively idempotent semiring. For example, Han and Li studied the properties of the adjoint matrix of incline matrices and present Crammer's rule [5]. Kim et al. [6] studied permanent theory for fuzzy square matrices and proved that $\mathrm{per}(A\,\mathrm{adj}(A)) = \mathrm{per}(A) = \mathrm{per}(\mathrm{adj}(A)A)$. This result was generalized to D_{01}-lattice matrices by Zhang [13]. In 2004, Duan [2] studied permanents and adjoint matrices of incline matrices and proved that A^n is equal to the adjoint matrix of A

if the matrix A satisfies $A \geq I_n$ and posed the following open problems:
Problem 1.1. Does the equality

$$\mathrm{per}(A\,\mathrm{adj}(A)) = \mathrm{per}((\mathrm{adj}(A))A) = \mathrm{per}(A)$$

hold for any square matrix A over an incline?
Problem 1.2. Does the equality

$$A^{n-1} = \mathrm{adj}(A)$$

hold for an $n \times n$ matrix A over an incline satisfying $a_{ii} \geq a_{jk}$ for all $i, j, k \in \underline{n}$?

In this paper, we will discuss permanents of matrices over a commutative additively idempotent semiring. In section 2, we give some properties and characterizations for permanents and obtain some inequalities for the permanents. In section 3, we consider the adjoint matrices of matrices over commutative additively idempotent semiring and answer Problem 1.1 and 1.2. Partial results obtained in this paper generalize the corresponding results on fuzzy matrices in [6, 10]on fuzzy matrices, on lattice matrices in [13] and on incline matrices in [2]. Finally, in section 4, we prove theorem 3.5 stated in section 3.

2 Properties and characterizations for permanents

In this paper, we always assume that R is a commutative additively semiring and $\leq$ be the canonical preorder realation induced by the + operation ($a \leq b$ if and only if $a + b = b$). Clearly, 0 is the least element in R, i.e., $0 \leq a$ for all $a \in R$. Moreover, for any $a, b, c, d \in R$,

$$\begin{array}{llcl} \text{(P1)} & a \leq b \text{ and } c \leq d & \Rightarrow & a + c \leq b + d \text{ and } ac \leq bd; \\ \text{(P2)} & a \leq b \text{ and } b \leq a & \Rightarrow & a = b. \end{array}$$

Let $M_n(R)$ be the set of all $n \times n$ matrices over R. Matrix operations on $M_n(R)$ are defined the same as that in a field. It is easy to verify that $(M_n(R), +, \cdot)$ is an additively idempotent semiring, thus the canonical preorder realation $\leq$ ($A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all $i, j \in \underline{n}$) on $M_n(R)$ satisfies (P1) and (P2). Denote by A^T the *transpose* of A.

Denote $\underline{n} = \{1, 2, \cdots, n\}$. For $A \in M_n(R)$, the *permanent* of A is defined as

$$\mathrm{per}(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)},$$

where S_n denotes the symmetric group of the set $\underline{n}$. Checking direct by the definition, we have the following proposition.

Proposition 2.1 *Let $A, B, C \in M_n(R)$ and $\lambda \in R$. Then*
(1) $\mathrm{per}(\lambda A) = \lambda^n \mathrm{per}(A)$, *where* $\lambda A = (\lambda a_{ij})_{n \times n}$*;*
(2) $\mathrm{per}(A^T) = \mathrm{per}(A)$*;*
(3) $\mathrm{per}(PAQ) = \mathrm{per}(A)$, *where P and Q are $n \times n$ permutation matrices;*

(4) if $a_{ii} \geq a_{ik}$ *for all* $i, k \in \underline{n}$*, then* $\mathrm{per}(A) = a_{11}a_{22}\cdots a_{nn}$*;*

(5) $\mathrm{per}\begin{pmatrix} A & C \\ O & B \end{pmatrix} = \mathrm{per}(A)\,\mathrm{per}(B)$*, where* O *denotes the* $n \times n$ *zero matrix;*

(6) if $A \leq B$*, then* $\mathrm{per}(A) \leq \mathrm{per}(B)$*.* □

For $k \in \underline{n-1}$, set $\Omega_{k,n} = \{\omega = (i_1, \ldots, i_k) \mid 1 < i_1 < \cdots < i_k < n\}$. For $\alpha = (i_1, \ldots, i_k), \beta = (j_1, \ldots, j_k) \in \Omega_{k,n}$ and $A \in M_n(R)$, we denote by $A[\alpha|\beta]$ the $k \times k$ submatrix of A whose $(u, v)-$entry is equal to $a_{i_u j_v}$ and by $A(\alpha|\beta)$ the $(n-k) \times (n-k)$ submatrix of A obtained from A by deleting rows α and columns β. Similar with the Laplace's theorem over a field, we have the following proposition.

Proposition 2.2 *For* $A \in M_n(R)$ *and* $\alpha \in \Omega_{k,n}$*, we have*

$$\mathrm{per}(A) = \sum_{\beta \in \Omega_{k,n}} \mathrm{per}(A[\alpha|\beta])\,\mathrm{per}(A(\alpha|\beta)). \tag{2.1}$$

In particularly, for $i \in \underline{n}$*, we have*

$$\mathrm{per}(A) = \sum_{j=1}^{n} a_{ij}\,\mathrm{per}(A(i|j)). \tag{2.2}$$

Proof. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. For any $\beta \in \Omega_{k,n}$, we have

$$\mathrm{per}(A[\alpha|\beta]) = \sum_{\sigma \in S_{\alpha,\beta}} \prod_{i=1}^{k} a_{\alpha_i \sigma(\alpha_i)},$$

where $S_{\alpha,\beta}$ is the set of all bijective mappings from α to β. Then, there are in total $k!$ terms in the expansion of $\mathrm{per}(A[\alpha|\beta])$. Similarly, there are in total $(n-k)!$ terms in the expansion of $\mathrm{per}(A(\alpha|\beta))$. Since the product of any term of $\mathrm{per}(A[\alpha|\beta])$ and any term of $\mathrm{per}(A(\alpha|\beta))$ is a term of $\mathrm{per}(A)$, there are in total $k!(n-k)!$ distinct terms in $\mathrm{per}(A[\alpha|\beta])\,\mathrm{per}(A(\alpha|\beta))$. Thus, there are in total $\binom{n}{k}k!(n-k)! = n!$ distinct terms in $\sum\limits_{\beta \in \Omega_{k,n}} \mathrm{per}(A[\alpha|\beta])\,\mathrm{per}(A(\alpha|\beta))$. Since any term of $\sum\limits_{\beta \in \Omega_{k,n}} \mathrm{per}(A[\alpha|\beta])\,\mathrm{per}(A(\alpha|\beta))$ is a term of $\mathrm{per}(A)$ and $\mathrm{per}(A)$ has $n!$ terms, we have

$$\mathrm{per}(A) = \sum_{\beta \in \Omega_{k,n}} \mathrm{per}(A[\alpha|\beta])\,\mathrm{per}(A(\alpha|\beta)),$$

as required. Taking $\alpha = (i)$ in (2.1), then we have

$$\mathrm{per}(A) = \sum_{j=1}^{n} a_{ij}\,\mathrm{per}(A(i|j)),$$

as required. This completes the proof. □

Proposition 2.3 *For* $A, B \in M_n(R)$*, we have*

$$\mathrm{per}(AB) \geq \mathrm{per}(A)\,\mathrm{per}(B). \tag{2.3}$$

Proof. Let T_n denote the set of all mappings from the set $\underline{n}$ to itself. By the definition of permanent, we have

$$\begin{aligned}
\operatorname{per}(AB) &= \sum_{\sigma\in S_n} (AB)_{1\sigma(1)}(AB)_{2\sigma(2)}\dots(AB)_{n\sigma(n)} \\
&= \sum_{\sigma\in S_n} (\sum_{1\le k\le n} a_{1k}b_{k\sigma(1)})(\sum_{1\le k\le n} a_{2k}b_{k\sigma(2)})\dots(\sum_{1\le k\le n} a_{nk}b_{k\sigma(n)}) \\
&= \sum_{\sigma\in S_n} \sum_{1\le k_1,k_2,\dots,k_n\le n} a_{1k_1}a_{2k_2}\cdots a_{nk_n}b_{k_1\sigma(1)}b_{k_2\sigma(2)}\dots b_{k_n\sigma(n)} \\
&= \sum_{\tau\in T_n} (a_{1\tau(1)}a_{2\tau(2)}\cdots a_{n\tau(n)} \sum_{\sigma\in S_n} b_{\tau(1)\sigma(1)}b_{\tau(2)\sigma(2)}\dots b_{\tau(n)\sigma(n)}) \\
&= \sum_{\tau\in S_n} (a_{1\tau(1)}a_{2\tau(2)}\cdots a_{n\tau(n)} \sum_{\sigma\in S_n} b_{\tau(1)\sigma(1)}b_{\tau(2)\sigma(2)}\dots b_{\tau(n)\sigma(n)}) \\
&\quad + \sum_{\tau\in T_n-S_n} (a_{1\tau(1)}a_{2\tau(2)}\cdots a_{n\tau(n)} \sum_{\sigma\in S_n} b_{\tau(1)\sigma(1)}b_{\tau(2)\sigma(2)}\dots b_{\tau(n)\sigma(n)}) \\
&\ge \sum_{\tau\in S_n} (a_{1\tau(1)}a_{2\tau(2)}\cdots a_{n\tau(n)} \sum_{\sigma\in S_n} b_{\tau(1)\sigma(1)}b_{\tau(2)\sigma(2)}\dots b_{\tau(n)\sigma(n)}) \\
&= \sum_{\tau\in S_n} (a_{1\tau(1)}a_{2\tau(2)}\cdots a_{n\tau(n)} \operatorname{per}(B)) \\
&= (\sum_{\tau\in S_n} a_{1\tau(1)}a_{2\tau(2)}\cdots a_{n\tau(n)}) \operatorname{per}(B) \\
&= \operatorname{per}(A)\operatorname{per}(B),
\end{aligned}$$

as required. This completes the proof. $\square$

Remark 2.4 *The inequality (2.3) is not always true on equal. For example, let $R = ([0,+\infty], \vee, \cdot)$, where $\vee = max$ and $\cdot$ is the usual multiplication of real numbers. It is easy to verify that R is a commutative additively idempotent semiring. Choose*

$$A = \begin{pmatrix} 1 & 0.5 \\ 2 & 2 \end{pmatrix}, \; B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix},$$

we have $\operatorname{per}(A) = 2, \operatorname{per}(B) = 1, \operatorname{per}(AB) = 4$, *so* $\operatorname{per}(AB) > \operatorname{per}(A)\operatorname{per}(B)$.

Corollary 2.5 *Let $A \in M_n(R)$. If A is idempotent (i.e. $A^2 = A$) with* $\operatorname{per}(A) \ge 1$, *then* $(\operatorname{per}(A))$ *is idempotent too.*

Proof. Since $A^2 = A$, using (2.3), we have $(\operatorname{per}(A))^2 \le \operatorname{per}(A)$. Since $\operatorname{per}(A) \ge 1$, we have $(\operatorname{per}(A))^2 \ge \operatorname{per}(A) \cdot 1 = \operatorname{per}(A)$. Thus we have $(\operatorname{per}(A))^2 = \operatorname{per}(A)$, as required. This completes the proof. $\square$

3 Adjoint matrix of a square matrix

In this section, we will discuss the adjoint matrix of a square matrix over a commutative additively idempotent semiring R and answer Problems 1.1 and 1.2. Recall that the *adjoin matrix* of $A \in M_n(R)$ is denoted by $\operatorname{adj}(A)$, which is defined

as an $n \times n$ matrix whose (i,j)th entry is $\mathrm{per}(A(j|i))$, where $A(j|i) \in M_{n-1}(R)$ obtained from A by deleting the jth row and the ith column.

Proposition 3.1 *For $A, B \in M_n(R)$, we have*

(1) $A \leq B$ implies $\mathrm{adj}(A) \leq \mathrm{adj}(B)$;

(2) $\mathrm{adj}(A) + \mathrm{adj}(B) \leq \mathrm{adj}(A+B)$;

(3) $(\mathrm{adj}(A))^T = \mathrm{adj}(A^T)$.

Proof. (1) It follows from the definition of the adjoin matrix and proposition 2.1(6).

(2) Since $A \leq A+B$ and $B \leq A+B$, using (1), we have $\mathrm{adj}(A) \leq \mathrm{adj}(A+B)$ and $\mathrm{adj}(B) \leq \mathrm{adj}(A+B)$, which implies $\mathrm{adj}(A) + \mathrm{adj}(B) \leq \mathrm{adj}(A+B)$.

(3) It is obvious. □

We say $A \in M_n(R)$ with $n \geq 2$ satisfies condition $(*)$ for convenient, if

$$a_{ii} \geq a_{jk} \text{ for all } i,j,k \in \underline{n}.$$

Clearly, if A satisfies condition $(*)$, then so is A^l for any positive integer l.

Lemma 3.2 *Let $A \in M_n(R)$ with $n \geq 2$. If A satisfies condition $(*)$, then for positive integer l, we have*

$$\mathrm{per}(A^l) = (\mathrm{per}\, A)^l.$$

Proof. Since $a_{ii} \geq a_{jk}$ for all $i,j,k \in \underline{n}$, we have $a_{11} = a_{22} = \cdots = a_{nn} \geq a_{ij}$ for $i,j \in \underline{n}$. Thus, we have $\mathrm{per}(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)} = a_{11}^n$. Denote by $a_{ij}^{(l)}$ the (i,j)th entry of A^l with $l \geq 2$, we have

$$a_{ij}^{(l)} = \sum_{j_1, j_2, \cdots, j_{l-1} \in \underline{n}} a_{ij_1} a_{j_1 j_2} \cdots a_{j_{l-1} j} \leq a_{11}^l$$

and

$$a_{ii}^{(l)} = \sum_{j_1, j_2, \cdots, j_{l-1} \in \underline{n}} a_{ij_1} a_{j_1 j_2} \cdots a_{j_{l-1} i} = a_{11}^l,$$

for $i,j \in \underline{n}$. Thus, we have $\mathrm{per}(A^l) = \sum_{\sigma \in S_n} (\ a_{1\sigma(1)}^{(l)} a_{2\sigma(2)}^{(l)} \ldots a_{n\sigma(n)}^{(l)}) = a_{11}^{ln} = (\mathrm{per}\, A)^l$, as required. This completes the proof. □

For a max-min $n \times n$ fuzzy matrix A, Thomason [12] proved that

$$\mathrm{adj}(A) = A^{n-1} \tag{3.1}$$

if A satisfies condition $(*)$. Duan [2] proved that (3.1) holds for matrices over distributive lattice with same condition and matrices over commutative incline under stronger condition. For matrices over a commutative additively idempotent semiring R, we have the following theorem.

Theorem 3.3 *Let $A \in M_n(R)$ with $n \geq 2$. If $a_{ii} \geq a_{jk}$ for all $i,j,k \in \underline{n}$, then*

(1) $\mathrm{adj}(A) = A^{n-1}$*;*

(2) $\mathrm{per}(\mathrm{adj}(A)) = (\mathrm{per}(A))^{n-1}$. (3.2)

Proof (1) First, we prove $\mathrm{per}(A(j|i)) \le a_{ij}^{(n-1)}$ (the (i,j)th entry of A^{n-1}). If $i = j$, by assumption, we have

$$\begin{aligned}
\mathrm{per}(A(i|i)) &= \sum_{\sigma\in S_n} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{i-1,\sigma(i-1)}a_{i+1,\sigma(i+1)}\cdots a_{n\sigma(n)} \\
&\le \sum_{\sigma\in S_n} a_{11}a_{22}\cdots a_{i-1,i-1}a_{i+1,i+1}\cdots a_{nn} \\
&= a_{11}a_{22}\cdots a_{i-1,i-1}a_{i+1,i+1}\cdots a_{nn} \\
&= a_{ii}^{n-1} \\
&\le a_{ii}^{(n-1)} (\text{because } a_{ii}^{n-1} \text{ is a term of } a_{ii}^{(n-1)}).
\end{aligned}$$

If $i \ne j$, we have

$$\mathrm{per}(A(j|i)) = \sum_{\substack{\sigma\in S_n \\ \sigma(j)=i}} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{j-1,\sigma(j-1)}a_{j+1,\sigma(j+1)}\cdots a_{n\sigma(n)}.$$

Let $T_\sigma = a_{1\sigma(1)}\cdots a_{j-1,\sigma(j-1)}a_{j+1,\sigma(j+1)}\cdots a_{n\sigma(n)}$ be any term of $\mathrm{per}(A(j|i))$. Since $\sigma(j) = i$, there exists a positive integer u such that $\sigma^u(i) = j$. Let t be the least positive integer such that $\sigma^t(i) = j$, then $i, \sigma(i), \ldots, \sigma^{t-1}(i), j$ are mutually different. Thus we have

$$T_\sigma \le a_{i\sigma(i)}a_{\sigma(i)\sigma^2(i)}\cdots a_{\sigma^{t-1}(i)j}a_{jj}^{n-1-t} \le a_{ij}^{(n-1)}.$$

By the choise of T_σ, we have $\mathrm{per}(A(j|i)) \le a_{ij}^{(n-1)}$, as required.

Next, we prove $a_{ij}^{(n-1)} \le \mathrm{per}(A(j|i))$. If $i = j$, then we have

$$\begin{aligned}
a_{ii}^{(n-1)} &= \sum_{j_1,j_2,\cdots,j_{n-2}\in\underline{n}} a_{ij_1}a_{j_1j_2}\cdots a_{j_{n-2}i} \\
&\le \sum_{j_1,j_2,\ldots,j_{n-2}\in\underline{n}} a_{11}a_{22}\cdots a_{i-1,i-1}a_{i+1,i+1}\cdots a_{nn} \\
&= a_{11}a_{22}\cdots a_{i-1,i-1}a_{i+1,i+1}\cdots a_{nn} \\
&\le \mathrm{per}(A(i|i)),
\end{aligned}$$

as required. If $i \ne j$, let $T = a_{ij_1}a_{j_1j_2}\cdots a_{j_{n-2}j}$ be any term of the expansion of $a_{ij}^{(n-1)}$. If $i, j_1, j_2, \ldots, j_{n-2}, j$ are mutually different, then T is a term of $\mathrm{per}(A(j|i))$, which implies $T \le \mathrm{per}(A(j|i))$. If $j_r = j_s$ for some $r, s \in \{0, 1, 2, \ldots, n-1\}$ with $r < s$ (taking $i = j_0$ and $j = j_{n-1}$), delecting $a_{j_rj_{r+1}}\cdots a_{j_{s-1}j_s}$ from T, we get $a_{ij_1}\cdots a_{j_{r-1}j_r}a_{j_sj_{s+1}}\cdots a_{j_{n-2}j}$. Repeating the above method until we can obtain $a_{il_1}a_{l_1l_2}\cdots a_{l_mj}$, where $0 \le m < n-2$ and $i, l_1, \ldots, l_m, j$ are mutually different. Let $\{p_1, p_2, \ldots, p_{n-m-2}\} = \underline{n} \setminus \{i, l_1, \ldots, l_m, j\}$, we have

$$T \le a_{il_1}a_{l_1l_2}\cdots a_{l_mj}a_{p_1p_1}a_{p_2p_2}\cdots a_{p_{n-m-2}p_{n-m-2}} \le \mathrm{per}(A(j|i)).$$

By the choice of T, we have $a_{ij}^{(n-1)} \leq \operatorname{per}(A(j|i))$, as required.

Consequently, $a_{ij}^{(n-1)} = \operatorname{per}(A(j|i))$ which implies $A^{n-1} = \operatorname{adj}(A)$.

(2) Using lemma 3.2, we have $\operatorname{per}(A^{n-1}) = (\operatorname{per} A)^{n-1}$. Thus, by (1), we have $\operatorname{per}(\operatorname{adj}(A)) = (\operatorname{per} A)^{n-1}$, as required. This completes the proof. □

Remark 3.4 Since any commutative incline is a commutative additively idempotent semiring, Theorem 3.3 solves Problem 1.2.

Wether the equality (3.2) in Theorem 3.3 is hold for any $n \times n$ matrix over an additively idempotent semiring is not known. Meanwhile it is well known that for an $n \times n$ matrix A over a field, $|\operatorname{adj}(A)| = |A|^{n-1}$, where $|A|$ is the determinant of A. Recall that $|A \operatorname{adj}(A)| = |A|^n$ for an $n \times n$ matrix A over a field. For a matrix over a commutative additively idempotent semiring R, we have the following theorem which will be proved in the following section.

Theorem 3.5 *For $A \in M_n(R)$ with $n \geq 2$, we have*

$$\operatorname{per}(A \operatorname{adj}(A)) = \operatorname{per}(\operatorname{adj}(A)A) = (\operatorname{per}(A))^n. \tag{3.3}$$

Remark 3.6 *Let $R = ([0,1], \vee, \cdot)$, where $\vee = \max$ and $\cdot$ is the usual multiplication of real numbers. Obviously, R is a commutative incline. Let*

$$A = \begin{pmatrix} 0.1 & 0 & 0.2 \\ 0 & 0.2 & 0.3 \\ 0 & 0 & 0.3 \end{pmatrix} \in M_3(R),$$

we have $\operatorname{per}(A) = 6 \cdot 10^{-3}$ *and* $\operatorname{per}(A \operatorname{adj}(A)) = 6^3 \cdot 10^{-9} = (\operatorname{per}(A))^3 \neq \operatorname{per}(A)$. *Since any commutative incline is a commutative additively idempotent semiring, Theorem 3.5 solves Problem 1.1 in the negative sense.*

Remark 3.7 *Since the fuzzy algebra $\mathbb{F} = ([0,1], \vee, \wedge)$ and any bounded distributive lattice are commutative additively idempotent semiring which satisfying the multiplicatin is idempotent, Theorem 3.5 generalizes Theorem 4 in [6] and Theorem 6 in [13].*

4 Proof of theorem 3.5

In the proof, we need the following notation and three lemmas.

Notation 4.1 (1) For $p, q \in \underline{n}, \sigma \in S_n$, denote

$$\Phi_{p,q}^{\sigma} = \{(i, \sigma(i)) | i \neq q, i \in \underline{n}\} \cup \{(p, \sigma(q))\}, \quad \Phi^{\sigma} = \Phi_{p,p}^{\sigma} = \{(i, \sigma(i)) | i \in \underline{n}\}.$$

(2) For $A \in M_n(R)$, denote by $A(p \Rightarrow q)$ the matrix obtained from A by replacing row q of A by row p of A.

Lemma 4.2 *Suppose $\sigma, \pi \in S_n$ and $p, q, r \in \underline{n}$ with $q \neq r$, then we have*

$$\Phi_{p,q}^{\sigma} \cup \Phi_{q,r}^{\pi} = \Phi^{\varphi} \cup \Phi_{p,r}^{\tau},$$

for some $\varphi, \tau \in S_n$.

Proof. We divide into two cases to define φ.

Case one: $q \neq (\pi^{-1}\sigma)^k(r)$ for any nonnegative integer k (we may consider σ^0 is the identical mapping of the set $\underline{n}$ for any $\sigma \in S_n$). In this case, define $\varphi : \underline{n} \to \underline{n}$ by

$$\varphi(i) = \begin{cases} \sigma(i) & \text{if } i \in U \\ \pi(i) & \text{if } i \in \underline{n} - U \end{cases}$$

where $U = \{(\pi^{-1}\sigma)^k(r) | k \in \mathbb{N}\}$ and $\mathbb{N}$ denotes the set of all nonnegative integers. Then, $\varphi \in S_n$ with $\varphi(q) = \pi(q)$ and $\varphi(r) = \sigma(r)$. In fact, for any $i, j \in \underline{n}$ with $i \neq j$, if $i, j \in U$ then $\sigma(i) \neq \sigma(j)$ which implies $\varphi(i) \neq \varphi(j)$, and similarly, if $i, j \in \underline{n} - U$ then $\varphi(i) \neq \varphi(j)$. We may suppose $i \in U$ and $j \in \underline{n} - U$. If $\varphi(i) = \varphi(j)$ then $\sigma(i) = \pi(j)$ and so $j = (\pi^{-1}\sigma)(i)$. Since $i = (\pi^{-1}\sigma)^k(r)$ for some $k \in \mathbb{N}$, we have $j = (\pi^{-1}\sigma)^{k+1}(r)$ and so $j \in U$, which is a contradiction. Then $\varphi(i) \neq \varphi(j)$. Therefore, the mapping φ is injective with $\underline{n}$ is a finite set, which implies $\varphi \in S_n$. It is clear that $\varphi(q) = \pi(q)$ and $\varphi(r) = \sigma(r)$ since $q \in \underline{n} - U$ and $r \in U$.

Case two: $q = (\pi^{-1}\sigma)^k(r)$ for some $k \in \mathbb{N}$. Let k_0 be the least nonnegative integer k such that $q = (\pi^{-1}\sigma)^k(r)$. Then $q = (\pi^{-1}\sigma)^{k_0}(r)$ and $k_0 \geq 1$ (because $q \neq r$). In this case, define $\varphi : \underline{n} \to \underline{n}$ by

$$\varphi(i) = \begin{cases} \pi(i) & \text{if } i \in U - (V \cup \{q\}) \\ \sigma(i) & \text{if } i \in (\underline{n} - U) \cup V \\ \pi(r) & \text{if } i = q \end{cases}$$

where $V = \{(\pi^{-1}\sigma)^t(r) | t = 0, 1, \cdots, k_0 - 1\} \subseteq U$ (Note that $q \in U - V$). Similarly, we can prove that $\varphi \in S_n$ with $\varphi(q) = \pi(r)$ and $\varphi(r) = \sigma(r)$.

By deleting the elements $(1, \varphi(1)), (2, \varphi(2)), \ldots, (n, \varphi(n))$ from the set $\Phi^{\sigma}_{p,q} \cup \Phi^{\pi}_{q,r}$, we can get the following n elements:

$$(1, b_1), \ldots, (q, b_q), \ldots, (p, b_r), \ldots, (n, b_n)$$

where $b_1, b_2, \ldots, b_n \in \underline{n}$ are mutually different, i.e., there exists a permutation $\tau \in S_n$ such that $b_j = \tau(j)$ for all $j \in \underline{n}$. By notation 4.1 (1), we have $\Phi^{\sigma}_{p,q} \cup \Phi^{\pi}_{q,r} = \Phi^{\varphi} \cup \Phi^{\tau}_{p,r}$ as required. This completes the proof. $\square$

Lemma 4.3 *Let* $A \in M_n(R)$ *and* $k \in \underline{n}$ *with* $n \geq 2$. *We have*

$$\mathrm{per}(A(p_1 \Rightarrow p_2))\,\mathrm{per}(A(p_2 \Rightarrow p_3)) \cdots \mathrm{per}(A(p_k \Rightarrow p_1)) \leq (\mathrm{per}(A))^k,$$

for $p_1, p_2, \ldots, p_k \in \underline{n}$.

Proof. We first prove that for $p, q, r \in \underline{n}$,

$$\mathrm{per}(A(p \Rightarrow q)) \cdot \mathrm{per}(A(q \Rightarrow r)) \leq \mathrm{per}(A) \cdot \mathrm{per}(A(p \Rightarrow r)). \tag{4.1}$$

In fact, the case $q = r$ is obvious by commutativity. Suppose $q \neq r$, then

$$\begin{aligned}
& \operatorname{per}(A(p \Rightarrow q)) \cdot \operatorname{per}(A(q \Rightarrow r)) \\
= & \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{p\sigma(q)} \cdots a_{n\sigma(n)} \cdot \sum_{\pi \in S_n} a_{1\pi(1)} \cdots a_{q\pi(r)} \cdots a_{n\pi(n)} \\
= & \sum_{\sigma, \pi \in S_n} a_{1\sigma(1)} \cdots a_{p\sigma(q)} \cdots a_{n\sigma(n)} a_{1\pi(1)} \cdots a_{q\pi(r)} \cdots a_{n\pi(n)}.
\end{aligned}$$

Denote by (i, j) the subscript of a_{ij}, we get the set $\Phi^{\sigma}_{p,q} \cup \Phi^{\pi}_{q,r}$ for

$$T := a_{1\sigma(1)} \cdots a_{p\sigma(q)} \cdots a_{n\sigma(n)} a_{1\pi(1)} \cdots a_{q\pi(r)} \cdots a_{n\pi(n)}.$$

By lemma 4.2 we have $\Phi^{\sigma}_{p,q} \cup \Phi^{\pi}_{q,r} = \Phi^{\varphi} \cup \Phi^{\tau}_{p,r}$ for some $\varphi, \tau \in S_n$. Therefore

$$T = a_{1\varphi(1)} a_{2\varphi(2)} \cdots a_{n\varphi(n)} a_{1\tau(1)} \cdots a_{p\tau(r)} \cdots a_{n\tau(n)}$$

is a term of $\operatorname{per}(A) \operatorname{per}(A(p \Rightarrow r))$, which implies $T \leq \operatorname{per}(A) \cdot \operatorname{per}(A(p \Rightarrow r))$. Since T is any term of $\operatorname{per}(A(p \Rightarrow q)) \cdot \operatorname{per}(A(q \Rightarrow r))$, we have (4.1).

Using (4.1), we have

$$\begin{aligned}
& \operatorname{per}(A(p_1 \Rightarrow p_2)) \operatorname{per}(A(p_2 \Rightarrow p_3)) \cdots \operatorname{per}(A(p_k \Rightarrow p_1)) \\
\leq & \operatorname{per}(A) \cdot \operatorname{per}(A(p_1 \Rightarrow p_3)) \operatorname{per}(A(p_3 \Rightarrow p_4)) \cdots \operatorname{per}(A(p_k \Rightarrow p_1)) \\
\leq & (\operatorname{per}(A))^2 \cdot \operatorname{per}(A(p_1 \Rightarrow p_4)) \operatorname{per}(A(p_4 \Rightarrow p_5)) \cdots \operatorname{per}(A(p_k \Rightarrow p_1)) \\
& \cdots\cdots\cdots\cdots \\
\leq & (\operatorname{per}(A))^{(k-1)} \cdot \operatorname{per}(A(p_1 \Rightarrow p_1)) \\
= & (\operatorname{per}(A))^k,
\end{aligned}$$

as required. This completes the proof. $\square$

Lemma 4.4 *For $A \in M_n(R)$ with $n \geq 2$, we have*

$$\operatorname{per}(A \operatorname{adj}(A)) \geq (\operatorname{per}(A))^n.$$

Proof. Denote by b_{ij} the (i, j)-th element of $A \operatorname{adj}(A)$, we have

$$b_{ij} = \sum_{k=1}^{n} a_{ik} \operatorname{per}(A(j|k)) = \operatorname{per}(A(i \Rightarrow j)).$$

Thus we have

$$\begin{aligned}
& \operatorname{per}(A \operatorname{adj}(A)) \\
= & \sum_{\pi \in S_n} \operatorname{per}(A(1 \Rightarrow \pi(1))) \operatorname{per}(A(2 \Rightarrow \pi(2))) \cdots \operatorname{per}(A(n \Rightarrow \pi(n))) \\
\geq & \operatorname{per}(A(1 \Rightarrow 1)) \operatorname{per}(A(2 \Rightarrow 2)) \cdots \operatorname{per}(A(n \Rightarrow n)) \\
= & (\operatorname{per}(A))^n,
\end{aligned}$$

as required. This completes the proof. $\square$

Proof of Theorem 3.5 By lemma 4.4, we have $\mathrm{per}(A\,\mathrm{adj}(A)) \geq (\mathrm{per}(A))^n$. In the following, we will prove $\mathrm{per}(A\,\mathrm{adj}(A)) \leq (\mathrm{per}(A))^n$.

Let $T := \mathrm{per}(A(1 \Rightarrow \pi(1)))\,\mathrm{per}(A(2 \Rightarrow \pi(2)))\cdots\mathrm{per}(A(n \Rightarrow \pi(n)))$ be any term of $\mathrm{per}(A\,\mathrm{adj}(A))$, where $\pi \in S_n$. Let $\pi = (i_1 i_2 \cdots i_r)(j_1 j_2 \cdots j_s)\cdots(l_1 l_2 \cdots l_u)$ be the decomposition of disjoint cycles with $r + s + \cdots + u = n$. Using lemma 4.3, we have

$$\begin{aligned}
T &= \prod_{k\in\underline{r}} \mathrm{per}(A(i_k \Rightarrow \pi(i_k))) \prod_{k\in\underline{s}} \mathrm{per}(A(j_k \Rightarrow \pi(j_k))) \cdots \prod_{k\in\underline{u}} \mathrm{per}(A(l_k \Rightarrow \pi(l_k))) \\
&= \Big(\prod_{k\in\underline{r-1}} \mathrm{per}(A(i_k \Rightarrow i_{k+1}))\Big)\,\mathrm{per}(A(i_r \Rightarrow i_1)) \cdot \Big(\prod_{k\in\underline{s-1}} \mathrm{per}(A(j_k \Rightarrow j_{k+1}))\Big) \\
&\quad \cdot \mathrm{per}(A(j_s \Rightarrow j_1)) \cdots \Big(\prod_{k\in\underline{u-1}} \mathrm{per}(A(l_k \Rightarrow l_{k+1}))\Big)\,\mathrm{per}(A(l_u \Rightarrow l_1)) \\
&\leq (\mathrm{per}(A))^r(\mathrm{per}(A))^s\cdots(\mathrm{per}(A))^u \\
&= (\mathrm{per}(A))^{r+s+\cdots+u} = (\mathrm{per}(A))^n.
\end{aligned}$$

By the choice of T, we have $\mathrm{per}(A\,\mathrm{adj}(A)) \leq (\mathrm{per}\,A)^n$ as required. Therefore, we have $\mathrm{per}(A\,\mathrm{adj}(A)) = (\mathrm{per}(A))^n$.

Finally, using proposition 2.1(2) and proposition 3.1(3), we have $\mathrm{per}(\mathrm{adj}(A)A) = (\mathrm{per}(A))^n$. This completes the proof. $\square$

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